Aspects of Classical Physics - Tutorial 6

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1 Charged Particle in a Magnetic Monopole Field

The Newton equation of motion for an electrically charged point particle (mass m, electric charge q, position $\vec{r}$, velocity $\vec{v} = \dot{\vec{r}}$) in the field of a magnetic monopole (charge g) located at the origin is as follows:

$$m\ddot{\vec{r}} = q\,\vec{v}\times\vec{B} = \kappa\,\vec{v}\times\frac{\vec{r}}{r^3} \quad \text{with} \quad \kappa = \frac{qg}{4\pi}\ .$$

a) Show that the kinetic energy $T = \frac{1}{2}m\vec{v}^2$ is conserved.

b) Compute the time change of the angular momentum $\vec{L} = m\,\vec{r}\times\vec{v}$. Use that to define a quantity $\vec{J}$ which is conserved. Interpret your result.
Hint: computing the time derivative of $\vec{e}_r = \frac{\vec{r}}{r}$ may be useful.

c) Show that $\vec{J}\cdot\vec{e}_r = -\kappa =$ constant. What is the geometric implication on the particle's trajectory?

d) Compute $\vec{J}^2$ and prove that $\vec{L}^2$ is conserved, too. What does that mean for the time evolution of $\vec{L}$? And what does the fact that $\vec{L}^2$ is constant imply for the trajectory of the particle?

e) Prove that $\frac{d^2}{dt^2}r^2 = \frac{d^2}{dt^2}(\vec{r}\cdot\vec{r}) = 2v^2 =$ constant. Solve for $r(t)$.

2 The modular group SL$(2,\mathbb{Z})$

In the next lecture you'll see that the group SL$(2,\mathbb{Z})$ appears naturally in the discussion of dyons, that is, monopoles with both electric and magnetic charges. Let's discuss some of its properties now. Define the projective action of SL$(2,\mathbb{Z})$ on the upper half of the complex plane $\mathbb{C} \ni \tau$ with Im$\tau > 0$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad a,b,c,d \in \mathbb{Z} \ \text{and}\ ad - bc = 1.$$

a) Show that this action preserves the upper half plane, that is, if the imaginary part of τ is positive, so is its image by any SL$(2,\mathbb{Z})$ transformation acting as above.

b) This action is not faithful, i.e., there are elements that act trivially. Show that the elements which act trivially are only $\{\mathbb{1}_2, -\mathbb{1}_2\}$. It follows that the action of the projective group PSL$(2,\mathbb{Z})$ = SL$(2,\mathbb{Z})/\{\pm\mathbb{1}_2\}$, in turn, is faithful.

c) Two very important special maps are $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$. Show that they indeed preserve the sign of Imτ. Show that they satisfy $S^2 = (ST)^3 = \text{id}$ (identity map).

d) Exhibit a pair of SL$(2,\mathbb{Z})$ matrices $\hat{S}$ and $\hat{T}$ corresponding to the maps S and T. Show that they satisfy the matrix identities $\hat{S}^2 = -\mathbb{1}_2$ and $(\hat{S}\hat{T})^3 = \mathbb{1}_2$ or $-\mathbb{1}_2$.

e) A fundamental region $\mathcal{F}$ is a closed subset of the upper half plane (including the real axis plus infinity) which contains exactly one point of each orbit under the PSL(2,$\mathbb{Z}$) action above. In other words, every point in the upper half plane can be mapped uniquely into $\mathcal{F}$. Try to identify such a region $\mathcal{F}$.

f) Show that $\hat{S}$ and $\hat{T}$ generate the whole group SL$(2,\mathbb{Z})$. This implies that S and T generate PSL$(2,\mathbb{Z})$.